

PURE-CYCLE HURWITZ FACTORIZATIONS AND MULTI-NODED ROOTED TREES

ROSENA R. X. DU AND FU LIU

ABSTRACT. Pure-cycle Hurwitz number counts the number of connected branched covers of the projective lines where each branch point has only one ramification point over it. The main result of the paper is that when the genus is 0 and one of the ramification indices is d , the degree of the covers, the pure-cycle Hurwitz number is d^{r-3} , where r is the number of branch points.

We approach this problem via the standard translation of Hurwitz numbers into group theory. We define a new class of combinatorial objects, multi-noded rooted trees, which generalize rooted trees. We find the cardinality of this new class which with proper parameters is exactly d^{r-2} . The main part of this paper is the proof that there is a bijection from factorizations of a d -cycle to multi-noded rooted trees via factorization graphs. This implies the desired formula.

1. INTRODUCTION

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition of d . We say a permutation $\sigma \in S_d$ has *cycle type* λ if $\lambda_1, \dots, \lambda_\ell$ are the lengths of the cycles in the cycle decomposition of σ . We call a permutation $\sigma \in S_d$ an e -cycle if its cycle type is $(e, 1, \dots, 1)$ for some $e \geq 2$. Given a permutation σ of cycle type λ , we define its *index* as $\iota(\sigma) = \iota(\lambda) = \sum_i (\lambda_i - 1)$.

Hurwitz numbers count the number of connected branched covers of the projective line with specified ramification. More precisely, given $d \geq 1$, $g \geq 0$, $r \geq 0$, and $\lambda^1, \dots, \lambda^r$ partitions of d , the *Hurwitz number* $h(d, r, g; \lambda^1, \dots, \lambda^r)$ counts the number of connected genus- g covers of the projective line of degree d with r branch points where the monodromy over the i th branch point has cycle type λ^i . If a cover has non-trivial automorphisms, we divide by the size of its automorphism group. According to the Riemann-Hurwitz formula, a branched cover satisfies

$$(1.1) \quad \sum_{i=1}^r \iota(\lambda^i) = 2d - 2 + 2g.$$

Therefore, we are only interested in data $(d, r, g; \lambda^1, \dots, \lambda^r)$ that satisfies the above formula.

There is also a group-theoretic description of Hurwitz numbers:

2010 *Mathematics Subject Classification.* 05A15.

Key words and phrases. Hurwitz number, multi-noded rooted tree, factorization graph.

The first author is partially supported by the National Science Foundation of China under Grant No. 10801053, Shanghai Rising-Star Program (No. 10QA1401900), and the Fundamental Research Funds for the Central Universities.

The second author is partially supported by the National Security Agency under Grant No. H98230-09-1-0029, and the National Science Foundation of China under Grant No. 10801053.

Definition 1.1. Assume (1.1). A *Hurwitz factorization of type* $(d, r, g; \lambda^1, \dots, \lambda^r)$ is a tuple $(\sigma_1, \dots, \sigma_r)$ satisfying:

- a) $\sigma_i \in S_d$ has cycle type λ^i ;
- b) $\sigma_1 \cdots \sigma_r = 1$;
- c) the σ_i 's generate a transitive subgroup of S_d .

Then the Hurwitz number $h(d, r, g; \lambda^1, \dots, \lambda^r)$ is the number of Hurwitz factorizations divided by $d!$. This interpretation will be the one we use in this paper.

There has been a lot of work on Hurwitz numbers. Most of it has studied situations where all but one or two branch points are simple; i.e., all but one or two λ^i 's have the form $(2, 1, \dots, 1)$. Hurwitz [5] and Goulden-Jackson [2] showed that if $\lambda^1 = \dots = \lambda^{r-1} = (2, 1, \dots, 1)$ and $\lambda^r = (\tau_1, \dots, \tau_n)$, then

$$(1.2) \quad h(d, r, 0; \lambda^1, \dots, \lambda^r) = \frac{(r-1)! d^{n-3} \prod_{i=1}^n \tau_i^{\tau_i} / \tau_i!}{m_1! m_2! \cdots m_d!},$$

where m_i is the number of i 's in λ^r for $1 \leq i \leq d$.

In this paper, we will study *pure-cycle* Hurwitz numbers. We say a Hurwitz number is *pure-cycle* if each λ^i is of the form $(e_i, 1, \dots, 1)$ for some integer $e_i \geq 2$. In other words, a pure-cycle Hurwitz number counts the number of genus- g covers of the projective line of degree d with r branch points where there is only one ramification point over each branch point, with ramification index e_i . In this situation, we will abbreviate our notation for the Hurwitz number to $h(d, r, g; e_1, \dots, e_r)$. Pure-cycle Hurwitz numbers were first studied in [7]. The authors showed that

$$(1.3) \quad h(d, 4, 0; e_1, e_2, e_3, e_4) = \min\{e_i(d+1-e_i)\}.$$

We consider the special case of pure-cycle Hurwitz numbers when the genus is 0 and one of the e_i 's is d . Since the order of e_i 's does not change the Hurwitz number, without loss of generality, we can assume $e_r = d$. Note that by the Riemann-Hurwitz formula (1.1), we must have $\sum_{i=1}^r (e_i - 1) = \sum_{i=1}^{r-1} (e_i - 1) + (d - 1) = 2d - 2 + 2 \cdot 0$. Hence, we require

$$(1.4) \quad \sum_{i=1}^{r-1} (e_i - 1) = d - 1.$$

Below is our main theorem:

Theorem 1.2. Suppose $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Then

$$(1.5) \quad h(d, r, 0; e_1, \dots, e_{r-1}, d) = d^{r-3}.$$

One checks that our theorem generalizes special cases of (1.2) and (1.3). Indeed, if $\lambda^r = (d)$ in (1.2), it becomes the pure-cycle Hurwitz number $h(d, r, 0; 2, \dots, 2, d)$. Then by the requirement (1.4), we have that $d = r$. Hence, (1.2) gives us

$$h(d, r, 0; 2, \dots, 2, d) = \frac{(d-1)! d^{1-3} d^d / d!}{0! \cdots 0! 1!} = d^{d-3} = d^{r-3}.$$

On the other hand, if we let $e_4 = d$ in (1.3), by simple calculus arguments, one sees that

$$h(d, r, 0; e_1, e_2, e_3, d) = d(d+1-d) = d^{4-3} = d^{r-3}.$$

We remark that since σ_r is a d -cycle, condition c) in the definition of Hurwitz factorization is automatically satisfied. Thus, to verify whether one tuple $(\sigma_1, \dots, \sigma_r)$ is a Hurwitz factorization of type $(d, r, 0; e_1, \dots, e_{r-1}, d)$, we only need to check

whether (a) and (b) are satisfied. Hence, we reduce our problem to a simpler combinatorial problem:

Definition 1.3. Fix a d -cycle τ . We say $(\sigma_1, \dots, \sigma_{r-1})$ is a *factorization of τ* the following conditions are satisfied:

- a) For each i , σ_i is a cycle in $S_{\text{supp}(\tau)}$;
- b) $\sigma_1 \cdots \sigma_{r-1} = \tau$.

If further for each i , σ_i is an e_i -cycle, we say $(\sigma_1, \dots, \sigma_{r-1})$ is a *factorization of τ of type (e_1, \dots, e_{r-1})* .

We denote by $\mathcal{Fac}(d, r, \tau; e_1, \dots, e_{r-1})$ the set of all the factorizations of τ of type (e_1, \dots, e_{r-1}) and $\text{fac}(d, r, \tau; e_1, \dots, e_{r-1})$ the cardinality of $\mathcal{Fac}(d, r, \tau; e_1, \dots, e_{r-1})$.

Clearly, the number of factorizations is independent of the choice of τ , so we often omit τ and just write $\text{fac}(d, r; e_1, \dots, e_{r-1})$.

If $(\sigma_1, \dots, \sigma_{r-1})$ is a factorization of τ of type (e_1, \dots, e_{r-1}) , one can show that

$$(1.6) \quad \sum_{i=1}^{r-1} (e_i - 1) = d - 1 + 2g, \text{ for some } g \geq 0.$$

In S_d , there are $(d-1)!$ permutations that are d -cycles. Thus, the number of Hurwitz factorizations of type $(d, r, g; e_1, \dots, e_{r-1}, d)$ is $(d-1)! \text{fac}(d, r; e_1, \dots, e_{r-1})$. Hence, we have that

$$h(d, r, g; e_1, \dots, e_{r-1}, d) = \frac{1}{d} \text{fac}(d, r; e_1, \dots, e_{r-1}).$$

We conclude that Theorem 1.2 is equivalent to the following theorem:

Theorem 1.4. Suppose $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Then

$$(1.7) \quad \text{fac}(d, r; e_1, \dots, e_{r-1}) = d^{r-2}.$$

We remark that if $e_1 = \dots = e_{r-1} = 2$, then $d = r$ and $\text{fac}(d, r; e_1, \dots, e_{r-1})$ counts the number of factorizations of a d -cycle into $d-1$ transpositions. According to our theorem, this number is

$$(1.8) \quad \text{fac}(d, d; 2, \dots, 2) = d^{d-2}.$$

Note that d^{d-2} also counts the number of labeled trees with d vertices. Different bijective proofs of (1.8) were given by Dénes [1], Moszkowski [8], Goulden-Pepper [3] and Goulden-Yong [4].

Different but equivalent versions of Theorem 1.4 have been studied. Given non-negative integers n_2, \dots, n_d , we say a factorization $(\sigma_1, \dots, \sigma_{r-1})$ of a d -cycle is of *cycle index (n_2, n_3, \dots, n_d)* if there are n_m m -cycles among $\sigma_1, \dots, \sigma_{r-1}$ for any $2 \leq m \leq d$. Note that with this definition, the condition (1.4) translates to

$$(1.9) \quad \sum_{m=2}^d (m-1)n_m = d-1.$$

Springer [9] and Irving [6] showed that assuming (1.9), the number of factorizations $(\sigma_1, \dots, \sigma_{r-1})$ of a d -cycle of cycle index (n_2, \dots, n_d) is given by

$$(1.10) \quad d^{r-2} \frac{(r-1)!}{\prod_{m=2}^d n_m!}.$$

Since the factorization number $\text{fac}(d, r; e_1, \dots, e_{r-1})$ we consider is invariant under order of e_i 's, we see that Theorem 1.4 is equivalent to their result. Springer [9] proved the result by symmetrizing the problem further. He gave a bijection between factorizations of cycle index (n_2, \dots, n_d) of *all* d -cycles in S_d and doubly-labeled oriented cacti preserving cycle lengths, then showed the latter class of combinatorial objects has cardinality $(d-1)!$ times (1.10). Irving's proof [6] is based on a bijection between factorizations of cycle index (n_2, \dots, n_d) of a *fixed* d -cycle and proper polymaps. (Irving's polymap is a generalization of the oriented cactus in [9]. It can be used in general factorization problems without the restriction that each σ_i has to be a cycle.)

The proofs given in [9, 6] can be considered as symmetrized bijective proofs of Theorem 1.4. We ask whether one can give a direct "de-symmetrized" bijective proof for it. The main purpose of this paper is to give such a bijection. In order to do that, we first construct a new class of combinatorial objects called *multi-noded rooted trees*, show that (with proper parameters) it has cardinality d^{r-2} , and then give a bijection between factorizations of a d -cycle and multi-noded rooted trees.

The plan of this article is as follows: In Section 2, we define multi-noded rooted trees and find its cardinality. In Section 3, we associated to each factorization a bipartite graph, which we call factorization graphs, and show this association is injective. In Section 4, we give characterizations of factorization graphs. Using this characterization, we show in Section 5 that there is a bijection between factorization graphs and multi-noded rooted trees, and then we conclude our theorems.

Acknowledgements. We would like to thank Brian Osserman for providing data on pure-cycle Hurwitz numbers and suggesting this problem to us. We are also grateful to Richard Stanley who pointed out to us the reference [9] and sent us a copy of it.

2. MULTI-NODED ROOTED TREES

We assume the readers are familiar with basic terminology in graph theory as presented in the appendix of [10]. We will review briefly the terms that will be used in this paper.

Recall that a graph is a pair (V, E) where V is the vertex set and $E \subseteq V \times V$ is the edge set of the graph. A *tree* is an acyclic graph, and a *rooted tree* is a tree with a special vertex, which we call the *root* of the given tree. Given a rooted tree T , let $e = \{v, w\}$ be an edge of T . If v is closer to the root than w , we call v the *parent* of w and w a *child* of v ; we also call v the *parent end* of e and w the *child end* of e .

We usually draw a rooted tree with its root at the top, put each child below the parent, and represent the vertices of the tree by distinct integers, i.e., $V \subseteq \mathbb{Z}$. In this paper, we always represent roots with the number 0. See Figure 2 for examples of rooted trees.

Suppose $S \subseteq \mathbb{Z}$ is a set of n elements and $0 \notin S$. Let \mathcal{R}_S be the set of rooted trees with vertex set $S \cup \{0\}$ and rooted at 0. It is well-known that

$$(2.1) \quad |\mathcal{R}_S| = (n+1)^{n-1}.$$

In this section, we will introduce a new class of combinatorial objects, called *multi-noded rooted trees*, which generalize \mathcal{R}_S , and we will find its cardinality, which is exactly d^{r-2} if we choose the right parameters.

Throughout this section, we assume $S = \{s_1 < s_2 < \dots < s_n\}$ is a set of n integers disjoint from $\{0\}$.

Definition 2.1. Suppose f_0, f_1, \dots, f_n are positive integers. We say $M = (T, \beta)$ is a *multi-noded rooted tree* on $S \cup \{0\}$ of vertex data (f_0, f_1, \dots, f_n) if $T = (S \cup \{0\}, E)$ is a rooted tree in \mathcal{R}_S and $\beta : E \rightarrow \mathbb{N}$ is a function satisfying that for any edge $e \in E$, if s_i is the parent end of e , then $\beta(e) \in \{1, 2, \dots, f_i\}$.

We define $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$ to be the set of all multi-noded rooted trees on $S \cup \{0\}$ of vertex data (f_0, f_1, \dots, f_n) .

We call the simple graph with one vertex and no edges *the trivial tree*.

Graph representations of multi-noded rooted trees. We give two ways to represent a multi-noded rooted tree $M = (T, \beta)$ graphically. The first way is to draw the rooted tree T and then label each edge e with $\beta(e)$. We call this the *edge-labeled representation* of M .

The second method is to draw a graph with *multi-noded* vertices: Given any positive integer f , an *f-noded vertex* is a picture of f nodes in a horizontal line and grouped together by a circle. (Note that the nodes in an f -noded vertex are considered to be ordered.) A *multi-noded vertex* is an f -noded vertex for some $f \in \mathbb{N}$. With this definition, we can draw $M = (T, \beta)$ in the following way:

- (1) For each $0 \leq i \leq n$, we draw an f_i -noded vertex which is labeled by s_i . These $n + 1$ multi-noded vertices are the vertices of M .
- (2) For any edge $e = \{s_i, s_j\}$ of T with s_i being the parent end of e , we connect the multi-noded vertex s_j to the $\beta(e)$ -th node in vertex s_i . These are the edges of M .

We call this the *multi-noded representation* of M .

Example 2.2. Let $T = T_1$ as shown in Figure 2. Suppose $M = (T, \beta)$ is the multi-noded rooted tree of vertex data $(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$ and $\beta(\{0, s_3\}) = 1$, $\beta(\{0, s_5\}) = 1$, $\beta(\{s_3, s_8\}) = 1$, $\beta(\{s_3, s_2\}) = 1$, $\beta(\{s_5, s_9\}) = 1$, $\beta(\{s_2, s_6\}) = 2$, $\beta(\{s_9, s_4\}) = 1$, $\beta(\{s_9, s_1\}) = 3$, $\beta(\{s_9, s_7\}) = 3$. The two representations of M are shown in Figure 1. Graph (a) is the edge-labeled representation and graph (b) is the multi-noded representation.

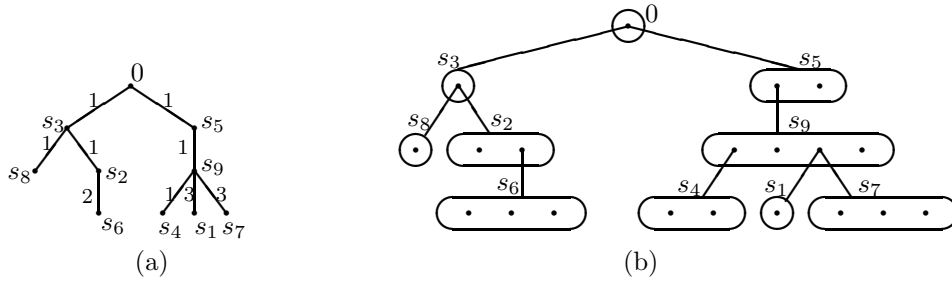


FIGURE 1. Two representations of a multi-noded rooted tree.

Remark 2.3. We remark that each of the two representations has its own advantage. The edge-labeled representation does not involve new combinatorial structure. We will use it to find the cardinality of $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$. The multi-noded representation contains the information of the vertex data when the edge-labeled representation does not. For example, graph (a) in Figure 1 could be the

graph of a multi-noded rooted tree of vertex data $(1, 1, 2, 1, 1, 1, 1, 1, 3)$ or anything bigger, but graph (a) in Figure 1 can only be associated with vertex data $(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$. The multi-noded representation will be used in a bijection we construct in Section 5.

Proposition 2.4. *The cardinality of $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$ is $\left(\sum_{j=0}^n f_j\right)^{n-1} f_0$.*

One sees that if $f_0 = f_1 = \dots = f_n = 1$, $\mathcal{MR}_S(f_0, f_1, \dots, f_{n-1})$ is in bijection with \mathcal{R}_S , and Proposition 2.4 recovers the result (2.1). One famous way to prove (2.1) is to construct the *Prüfer sequence*. In fact, we will use this idea to prove Proposition 2.4. Therefore, we will first review the construction of Prüfer sequences.

Prüfer sequences. Given a rooted tree $T \in \mathcal{R}_S$, we define a sequence T_1, T_2, \dots, T_{n+1} of subtrees of T as follows: Set $T_1 = T$. If $i < n + 1$ and T_i has been defined, then define T_{i+1} to be the tree obtained from T_i by removing its largest leaf v_i and the edge e_i incident to v_i . Then define w_i to be the other end of e_i , (i.e. w_i is the parent of v_i), and let $\gamma(T) := (w_1, w_2, \dots, w_n)$. We call $\gamma(T)$ the *Prüfer sequence* of T .

It is clear that $w_i \in S \cup \{0\}$ for $1 \leq i \leq n - 1$ and $w_n = 0$. Hence, $\gamma(T) \in (S \cup \{0\})^{n-1} \times \{0\}$. The proof of that γ is a bijection from \mathcal{R}_S to $(S \cup \{0\})^{n-1} \times \{0\}$ can be found in many places in the literature, for example, see [11, Page 25].

Example 2.5. Let T be the first tree shown in Figure 2. Then T_1, T_2, T_3 and T_4 in Figure 2 are the first four trees appeared in the construction of the Prüfer sequence of T . Continuing this construction, we obtain $\gamma(T) = (s_3, s_9, s_2, s_9, s_3, 0, s_9, s_5, 0)$.

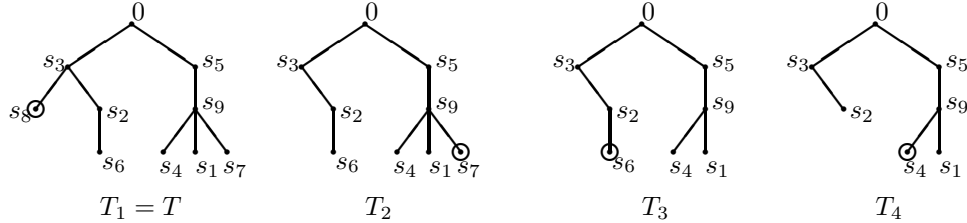


FIGURE 2. Constructing the Prüfer sequence of a rooted tree.

Proof of Proposition 2.4. For convenience, we write $s_0 := 0$. We denote by H the set of matrices $\begin{pmatrix} w_1 & w_2 & \dots & w_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$ satisfying $(w_i, b_i) \in \bigcup_{j=0}^n \{(s_j, k) \mid 1 \leq k \leq f_j\}$ for any $1 \leq i \leq n - 1$ and $(w_n, b_n) \in \{(0, k) \mid 1 \leq k \leq f_0\}$. Since $\bigcup_{j=0}^n \{(s_j, k) \mid 1 \leq k \leq f_j\}$ has cardinality $\sum_{j=0}^n f_j$ and $\{(0, k) \mid 1 \leq k \leq f_0\}$ has cardinality f_0 , the cardinality of H is $\left(\sum_{j=0}^n f_j\right)^{n-1} f_0$. Our goal is to show that there is a bijection between $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$ and H .

We will use the above algorithm for obtaining Prüfer sequences of rooted trees to define this bijection.

Suppose $M = (T, \beta) \in \mathcal{MR}_S(f_0, f_1, \dots, f_n)$. Let $\gamma(T) = (w_1, \dots, w_n)$ be the Prüfer sequence of T and e_1, \dots, e_n the edges removed in the procedure. We set

$b_i := \beta(e_i)$. (In the labeled-edge representation of G , b_i is the label of the edge e_i that is removed at step i .) Let

$$\tilde{\gamma}(M) = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}.$$

One sees that $\tilde{\gamma}(G) \in H$. Hence, $\tilde{\gamma}$ is a map from $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$ to H .

On the other hand, suppose $\begin{pmatrix} w_1 & w_2 & \cdots & w_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$ is H . Then $(w_1, \dots, w_n) \in (S \cup \{0\})^{n-1} \times \{0\}$. Since γ give a bijection between \mathcal{R}_S and $(S \cup \{0\})^{n-1} \times \{0\}$, we have $\gamma^{-1}(w_1, \dots, w_n) \in \mathcal{R}_S$. Let $T := \gamma^{-1}(w_1, \dots, w_n)$. We can apply the algorithm to obtain the Prüfer sequence of T and record the order of the edges that were deleted. We then label the edge that was removed in the i th step with number b_i . This procedure give us a rooted tree T with labeled edges, which is the edge-labeled representation of a multi-noded rooted tree $M = (T, \beta)$. One can check this procedure is the inverse of $\tilde{\gamma}$.

Therefore, $\tilde{\gamma}$ is a bijection between $\mathcal{MR}_S(f_0, f_1, \dots, f_n)$ and H . Thus, the conclusion follows. \square

Example 2.6. For the multi-noded rooted tree M in Example 2.2, we have

$$\tilde{\gamma}(G) = \begin{pmatrix} s_3 & s_9 & s_2 & s_9 & s_3 & 0 & s_9 & s_5 & 0 \\ 1 & 3 & 2 & 1 & 1 & 1 & 3 & 1 & 1 \end{pmatrix}.$$

Corollary 2.7. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then the cardinality of $\mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$ is d^{r-2} .

Proof. Let $n := r - 1$ and $f_i := e_i - 1$ for any $1 \leq i \leq n = r - 1$. Then by Proposition 2.4, we have

$$|\mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)| = |\mathcal{MR}_S(1, f_1, \dots, f_n)| = (1 + \sum_{i=1}^n f_i)^{n-1} = d^{r-2}. \quad \square$$

Corollary 2.7 provides us with a class of objects with cardinality d^{r-2} , which is the cardinality arising in Theorem 1.4. The rest of the paper is devoted to finding a bijection between multi-noded rooted trees of vertex data $(1, e_1 - 1, \dots, e_{r-1} - 1)$ and factorizations of a d -cycle of type (e_1, \dots, e_{r-1}) .

We finish this section with a definition, which will be used in Section 5.

Definition 2.8. Suppose f_0, \dots, f_n are positive integers and let $d = \sum_{i=0}^n f_i$. We say (M, \mathfrak{l}) is a *labeled multi-noded rooted tree of vertex data* (f_0, f_1, \dots, f_n) if $M \in \mathcal{MR}_S(f_0, f_1, \dots, f_n)$ and \mathfrak{l} is a labeling of the nodes of M with set $[d]$. (So \mathfrak{l} is a bijection from the set of the nodes of M to $[d]$.)

We denote by $\mathcal{LMR}(f_0, f_1, \dots, f_n)$ the set of all the labeled multi-noded rooted trees of vertex data (f_0, f_1, \dots, f_n) .

3. GRAPHS ASSOCIATED TO FACTORIZATIONS

Let $\tau \in S_d$ be a d -cycle, and e_1, \dots, e_{r-1} integers no less than 2. Let $S = \{s_1 < s_2 < \dots < s_{r-1}\}$ be a set of integers disjoint from $\{0, 1, 2, \dots, d\}$. For any cycle $\gamma \in S_d$, we denote by \mathbf{C}_γ the circle with nodes labeled by numbers in γ in clockwise order.

In this section, we associate a bipartite graph to each factorization of τ of type (e_1, \dots, e_{r-1}) . By discussing some properties of these graphs, we show that with the restriction $\sum_{j=1}^r (e_j - 1) = d - 1$ this association is an injection from $\mathcal{Fac}(d, r, \tau, e_1, \dots, e_{r-1})$ to its image set and thus is a bijection.

Definition 3.1. We call a graph G an S - $[d]$ bipartite graph if the vertex set of G is $S \cup [d]$ and any edge of G connects a vertex in S to a vertex in $[d]$. For any vertex v in an S - $[d]$ bipartite graph G or any subgraph of G , we call it an S -vertex if it is in S and a $[d]$ -vertex otherwise.

We denote by $\mathcal{G}_S(d, r; e_1, \dots, e_{r-1})$ the set of all S - $[d]$ bipartite graphs G satisfying for each $j : 1 \leq j \leq r - 1$ the vertex s_j has degree e_j .

S - $[d]$ bipartite Graph associated to factorizations. Suppose (1.1) and $(\sigma_1, \dots, \sigma_{r-1})$ is a factorization of τ of type (e_1, \dots, e_{r-1}) . We associate to $(\sigma_1, \dots, \sigma_{r-1})$ a graph $G = (V, E)$ with vertex set $V = S \cup \text{supp}(\tau) = S \cup [d]$ and edge set E consisting of all the pairs $\{s_j, \nu\}$ where $\nu \in \text{supp}(\sigma_j)$. We call G a *factorization graph* of type $(d, r, \tau; e_1, \dots, e_{r-1})$.

Example 3.2. Let $d = 20$, $r = 10$, $\tau = (1 \ 2 \ \dots \ 20)$ and $\sigma_1 = (10 \ 11)$, $\sigma_2 = (14 \ 15 \ 19)$, $\sigma_3 = (1 \ 19)$, $\sigma_4 = (3 \ 4 \ 5)$, $\sigma_5 = (1 \ 2 \ 13)$, $\sigma_6 = (15 \ 16 \ 17 \ 18)$, $\sigma_7 = (7 \ 8 \ 9 \ 11)$, $\sigma_8 = (19 \ 20)$ and $\sigma_9 = (2 \ 5 \ 6 \ 11 \ 12)$. One verifies that $(\sigma_1, \sigma_2, \dots, \sigma_9)$ is a factorization of τ of type $(2, 3, 2, 3, 3, 4, 4, 2, 5)$. The corresponding factorization graph is shown in Figure 3. (Note that the bipartite graph in the figure is drawn in a special way such that the $[d]$ -vertices are embedded onto \mathbf{C}_τ . It will become clear later why we draw the graph this way.)

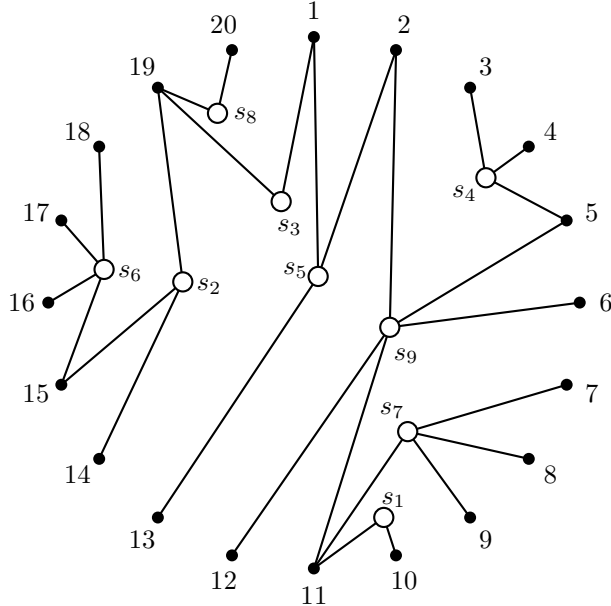


FIGURE 3. A factorization graph G of type $(20, 10, (1 \ 2 \ \dots \ 20); 2, 3, 2, 3, 3, 4, 4, 2, 5)$.

We denote by $\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$ the set of all the factorization graphs of type $(d, r, \tau; e_1, \dots, e_{r-1})$. Clearly

$$\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1}) \subset \mathcal{G}_S(d, r; e_1, \dots, e_{r-1}).$$

One may notice that the factorization graph in Figure 3 is a tree. In fact this is not a coincidence. The following lemma and corollary discuss conditions when G is a tree.

Lemma 3.3. *Suppose $G \in \mathcal{G}_S(d, r; e_1, \dots, e_{r-1})$ is connected. Then $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ if and only if G is a tree.*

Proof. Any graph is a tree if and only if the graph is connected and the number of vertices is one more than the number of edges. Therefore, G is a tree if and only if $1 = |S \cup \text{supp}(\tau)| - \sum_{j=1}^{r-1} e_j = r - 1 + d - \sum_{j=1}^{r-1} e_j$, which is equivalent to $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. \square

Corollary 3.4. *Suppose $G \in \mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$. Then $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ if and only if G is a tree.*

Proof. Suppose G is the factorization graph associated to $(\sigma_1, \dots, \sigma_{r-1})$, a factorization of τ . Since τ is a d -cycle, one sees $(\sigma_1, \dots, \sigma_{r-1})$ generates a transitive subgroup of S_d . Thus, any two $[d]$ -vertices of G are connected by a path. However, any S -vertex is connected to some $[d]$ -vertex. Hence, G is connected. Then the conclusion follows from Lemma 3.3. \square

It is not obvious from the definition that any two different factorizations of τ have different factorization graphs. We will show this is true at the end of this section by induction on r , and to achieve this, we discuss conditions on factorizations of τ .

Lemma 3.5. *Suppose $\mu = (u_1, \dots, u_q)$ is a q -cycle and $\eta \in S_{\text{supp}(\mu)}$ satisfying $\text{supp}(\eta) = \{u_{j_1}, \dots, u_{j_p}\} \subseteq \{u_1, u_2, \dots, u_q\}$ for some $j_1 > \dots > j_p$. Let s be the number of disjoint cycles (including the ones of length 1) in the cycle decomposition of $\mu\eta$. Then $s \leq p$, and the followings are equivalent:*

- (i) $s = p$.
- (ii) $\eta = (u_{j_1}, \dots, u_{j_p})$.
- (iii) $\mu\eta = (u_{j_1+1}, u_{j_1+2}, \dots, u_q, u_1, \dots, u_{j_p})(u_{j_p+1}, u_{j_p+2}, \dots, u_{j_{p-1}}) \cdots (u_{j_2+1}, u_{j_2+2}, \dots, u_{j_1})$.

Remark 3.6. In this paper, whenever we talk about cycle decomposition, in addition to the disjoint cycles of length greater than 1 appearing in the standard cycle decomposition of a permutation, we also include “cycles” of length 1. By convention, each of these contains exactly one fixed point of the permutation. We consider the support of each “1-cycle” to be its associated fixed point. Thus, the support of the cycles in the cycle decomposition of a permutation in S_d always gives a partition of $[d]$.

Proof of Lemma 3.5. Clearly, if $u_i \notin \text{supp}(\eta)$, then $\mu(u_i) = \mu\eta(u_i)$. Hence, under the permutation $\mu\eta$, we must have

$$\begin{aligned} u_{j_1+1} &\mapsto u_{j_1+2} \mapsto \cdots \mapsto u_q \mapsto u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{j_p-1} \mapsto u_{j_p} \\ u_{j_p+1} &\mapsto u_{j_p+2} \mapsto \cdots \mapsto u_{j_{p-1}}, \\ u_{j_{p-1}+1} &\mapsto u_{j_{p-1}+2} \mapsto \cdots \mapsto u_{j_{p-2}}, \\ &\dots \\ u_{j_2+1} &\mapsto u_{j_2+2} \mapsto \cdots \mapsto u_{j_1}. \end{aligned}$$

Hence, the numbers in each line have to be in the same cycle in the cycle decomposition of $\mu\eta$. Therefore the number of disjoint cycles in $\mu\eta$ is at most the number of lines we have above, i.e., $s \leq p$.

It is easy to verify that (ii) and (iii) are equivalent. We show that (i) is equivalent to (iii). We have $s = p$ if and only if the number at the end of each line is mapped to the number at the front under $\mu\eta$. This means

$$\mu\eta(u_{j_p}) = u_{j_1+1}, \mu\eta(u_{j_{p-1}}) = u_{j_p+1}, \dots, \mu\eta(u_{j_1}) = u_{j_2+1},$$

i.e.,

$$\begin{aligned} \eta(u_{j_p}) &= \mu^{-1}(u_{j_1+1}) = u_{j_1}, \eta(u_{j_{p-1}}) = \mu^{-1}(u_{j_p+1}) = u_{j_p}, \dots \\ &\dots, \eta(u_{j_1}) = \mu^{-1}(u_{j_2+1}) = u_{j_2}. \end{aligned}$$

Then our conclusion follows. \square

Remark 3.7. We can also understand Lemma 3.5 combinatorially: Suppose $\mu = (u_1, \dots, u_q)$ is a q -cycle and $\eta \in S_{\text{supp}(\mu)}$ satisfying $\text{supp}(\eta) = \{u_{j_1}, \dots, u_{j_p}\} \subseteq \{u_1, u_2, \dots, u_q\}$ and $j_1 > \cdots > j_p$. Recall \mathbf{C}_μ is a circle whose nodes are labeled by u_1, \dots, u_q in clockwise order.

Then the followings are equivalent:

- (i) There are p cycles in the cycle decomposition of $\mu\eta$.
- (ii) The numbers in η appear counterclockwise on \mathbf{C}_μ .
- (iii) We can cut \mathbf{C}_μ into consecutive pieces such that each piece forms a cycle in the cycle decomposition of $\mu\eta$ when reading clockwise.

Example 3.8. Let $\mu = \tau = (1 \ 2 \ \cdots \ 20)$ and $\eta = \sigma_9^{-1} = (12 \ 11 \ 6 \ 5 \ 2)$ as in Example 3.2. We have

$$\mu\eta = \tau\sigma_9^{-1} = (1 \ 2 \ \cdots \ 20)(12 \ 11 \ 6 \ 5 \ 2) = (3 \ 4 \ 5)(7 \ 8 \ 9 \ 10 \ 11)(13 \ 14 \ \cdots \ 20 \ 1 \ 2)(6)(12).$$

See Figure 4. Clearly 12, 11, 6, 5, 2 appear counterclockwise on \mathbf{C}_τ . If we cut \mathbf{C}_τ after each of 2, 5, 6, 11, 12, then we get exactly 5 consecutive pieces (3, 4, 5), (6), (7, 8, 9, 10, 11), (12) and (13, 14, \dots, 20, 1, 2) when reading the numbers in clockwise order.

Lemma 3.9. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$. Then G is a tree by Corollary 3.4.

Suppose by deleting s_{r-1} and its incident edges from G , we obtain trees $Q_1, \dots, Q_k, Q_{k+1}, \dots, Q_{e_{r-1}}$, where for $1 \leq i \leq k$, the $[d]$ -vertex set of Q_i has size m_i for some $m_i \geq 2$, and for $k+1 \leq i \leq e_{r-1}$, Q_i just contains one single $[d]$ -vertex.

For any $i : 1 \leq i \leq k$, let B_i be the set of j for which s_j is in Q_i . Then $\{B_1, \dots, B_k\}$ is a partition of $[r-2] := \{1, 2, \dots, r-2\}$.

For $1 \leq i \leq k$, let $\gamma_i := \prod_{j \in B_i} \sigma_j$ and for $k+1 \leq i \leq e_{r-1}$, let γ_i be the 1-cycle containing the only $[d]$ -vertex of Q_i . Then

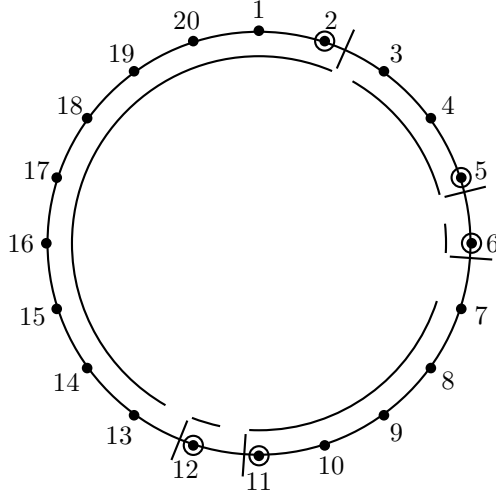


FIGURE 4. The products of two cycles $\mu = (1\ 2\ \cdots\ 20)$ and $\eta = (12\ 11\ 6\ 5\ 2)$.

- (i) $\gamma_1 \cdots \gamma_k \gamma_{k+1} \cdots \gamma_{e_{r-1}}$ is the cycle decomposition of $\sigma_1 \cdots \sigma_{r-2} = \tau \sigma_{r-1}^{-1}$.
- (ii) γ_i is an m_i -cycle on the $[d]$ -vertex set of Q_i .
- (iii) $(\sigma_j)_{j \in B_i}$ is a factorization of γ_i , and Q_i is the factorization graph associated to this factorization.
- (iv) $\sum_{j \in B_i} (e_j - 1) = m_i - 1$.

Example 3.10. Let $d, r, \sigma_1, \dots, \sigma_9, \tau$ and G be defined as in Example 3.2. So G is the graph in Figure 3. If we delete s_9 and its incident edges from G , we obtain $e_9 = 5$ trees, including two trees that are only a single $[d]$ -vertex. Let Q_1, Q_2, Q_3, Q_4 and Q_5 denote the five trees with $[d]$ -vertex set $\{3, 4, 5\}$, $\{7, 8, \dots, 11\}$, $\{13, 14, \dots, 20, 1, 2\}$, $\{6\}$ and $\{12\}$ respectively. Using the notation of Lemma 3.9, we have $k = 3$, $m_1 = 3$, $m_2 = 5$, $m_3 = 10$, and the corresponding partition of $[r - 2] = [8]$ is $B_1 = \{4\}$, $B_2 = \{1, 7\}$, $B_3 = \{2, 3, 5, 6, 8\}$. Let $\gamma_1 = \sigma_4 = (3\ 4\ 5)$, $\gamma_2 = \sigma_1 \sigma_7 = (7\ 8\ \cdots\ 11)$, $\gamma_3 = \sigma_2 \sigma_3 \sigma_5 \sigma_6 \sigma_8 = (13\ 14\ \cdots\ 20\ 1\ 2)$, $\gamma_4 = (6)$ and $\gamma_5 = (12)$. One can check that $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5$ is the cycle decomposition of $\tau \sigma_9^{-1}$, and for each $i : 1 \leq i \leq 3$, (ii), (iii) and (iv) hold.

Proof of Lemma 3.9. Since all the e_j 's are greater than 1, we have that any Q_i for $k + 1 \leq i \leq e_{r-1}$ does not contain any S -vertices. Therefore, each s_j for any $j \in [r - 2]$ is in one of Q_1, \dots, Q_k . Thus, $\{B_1, \dots, B_k\}$ is a partition of $[r - 2]$. Let $i \in \{1, \dots, k\}$ and $j \in B_i$. One sees that all the $[d]$ -vertices incident to s_j have to be in Q_i as well. Therefore, $\text{supp}(\sigma_j)$ is contained in the $[d]$ -vertex set of Q_i . Thus, for any $j_1 \in B_{i_1}$ and $j_2 \in B_{i_2}$ with $i_1 \neq i_2$, we have that $\text{supp}(\sigma_{j_1})$ and $\text{supp}(\sigma_{j_2})$ are disjoint, which implies that $\sigma_{j_1} \sigma_{j_2} = \sigma_{j_2} \sigma_{j_1}$. Hence,

$$\prod_{i=1}^k \gamma_i = \prod_{i=1}^k \prod_{j \in B_i} \sigma_j = \prod_{j=1}^{r-2} \sigma_j = \tau \sigma_{r-1}^{-1}.$$

Furthermore, for each $i : 1 \leq i \leq k$, $\gamma_i = \prod_{j \in B_i} \sigma_j$ is a permutation on the $[d]$ -vertex set of Q_i . Therefore, the support of γ_i 's ($1 \leq i \leq e_{r-1}$) are completely disjoint. Hence, we can partition the cycles in the cycle decomposition of $\tau \sigma_{r-1}^{-1}$

into e_{r-1} groups such that the product of the i th group of cycles is exactly γ_i . This implies that e_{r-1} is no greater than the number of cycles in the cycle decomposition of $\tau\sigma_{r-1}^{-1}$. However, by applying Lemma 3.5 with $\mu = \tau$ and $\eta = \sigma_{r-1}^{-1}$, we have that the number of cycles in the cycle decomposition of $\tau\sigma_{r-1}^{-1}$ is no greater than e_{r-1} . Hence, these two numbers are equal. So each γ_i is one cycle in the cycle decomposition of $\tau\sigma_{r-1}^{-1}$. We conclude (i),(ii) and (iii). Finally, (iv) follows from (iii) and Corollary 3.4. \square

Combining Lemma 3.9, Lemma 3.5 and Remark 3.7, we have the following corollary.

Corollary 3.11. *Suppose $\sum_{j=1}^{r-1}(e_j-1) = d-1$ and $(\sigma_1, \dots, \sigma_{r-1})$ is a factorization of τ of type (e_1, \dots, e_{r-1}) . Then we have the following conclusions:*

- (i) *For each $j : 1 \leq j \leq r-1$, the numbers in σ_j appear clockwise on \mathbf{C}_τ .*
- (ii) *Let $\gamma_1, \dots, \gamma_k, \gamma_{k+1}, \dots, \gamma_{e_{r-1}}$ be defined as in Lemma 3.9. Then $\text{supp}(\gamma_1), \dots, \text{supp}(\gamma_{e_{r-1}})$ partition \mathbf{C}_τ into consecutive pieces. Furthermore, for each $i : 1 \leq i \leq e_{r-1}$, the numbers in γ_i appear consecutively on \mathbf{C}_τ reading clockwise. Moreover, each γ_i contains exactly one number from σ_{r-1} and this number is the last number appearing on \mathbf{C}_τ .*

Proof. By Lemma 3.9, we have that the number of cycles in the cycle decomposition of $\tau\sigma_{r-1}^{-1}$ is equal to e_{r-1} , the size of the support of σ_{r-1}^{-1} . Hence, by Lemma 3.5 and Remark 3.7, we have (ii) and the numbers in σ_{r-1} appear clockwise on \mathbf{C}_τ . We can conclude (i) for other j 's by applying Lemma 3.9/(iii)(iv), Lemma 3.5 and Remark 3.7 recursively. \square

By Corollary 3.11/(i), one sees that with the condition $\sum_{j=1}^{r-1}(e_j-1) = d-1$, no two different factorizations of τ can have the same factorization graph.

Corollary 3.12. *Suppose $\sum_{j=1}^{r-1}(e_j-1) = d-1$. The way we associate a graph to a factorization gives a bijection between the set $\mathcal{F}ac(d, r, \tau; e_1, \dots, e_{r-1})$ and the set $\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$.*

Remark 3.13. We remark that if $e_1 = \dots = e_{r-1} = 2$, then $d = r$ and $\mathcal{F}ac(d, d, \tau; 2, \dots, 2)$ contains factorizations of a d -cycle τ into $d-1$ transpositions. In this case for any $G \in \mathcal{G}_S^*(d, d, \tau; 2, 2, \dots, 2)$, the S -vertices of G have degree 2. For each S -vertex $s_j \in G$, suppose s_j is incident to ν_{j_1} and ν_{j_2} . We can replace s_j and its two incident edges by one edge connecting ν_{j_1} and ν_{j_2} . Then we get a tree on vertex set $[d]$. Therefore, the bijection discussed in Corollary 3.12 becomes a bijection between trees on d vertices and factorizations of a d -cycle into $d-1$ transpositions, which is the same as the bijection defined by Moszkowski in [8] and the *circle chord diagram* construction defined by Goulden and Yong in [4].

4. CHARACTERIZATION OF FACTORIZATION GRAPHS

In this section, we will give a proposition (Proposition 4.4) to characterize properties of graphs in $\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$, which will be used to construct bijections between factorization graphs and multi-noded rooted trees. We first give definitions that are useful for the statement of the proposition.

Definition 4.1. Suppose $S' \subseteq S$ and γ is a cycle in S_d . Let G be an S' - $\text{supp}(\gamma)$ bipartite tree.

Suppose $s \in S'$. We say s has the *consecutive partition property* (or *CPP*) on (G, γ) if after we remove s and all its incident edges from G , the sets of $[d]$ -vertices of the subtrees we obtain partition the circle \mathbf{C}_γ into consecutive pieces.

Suppose $\nu \in \text{supp}(\gamma)$ and $\{s_{j_1} < s_{j_2} < \dots < s_{j_t}\}$ are the set of S -vertices incident to ν in G . By removing ν and all its incident edges, suppose we obtain t subtrees. We say ν has the *counterclockwise increasing consecutive partition property* (or *CICPP*) on (G, γ) if the following are satisfied:

- a) The $[d]$ -vertices of the t subtrees partition $\mathbf{C}_\gamma \setminus \{\nu\}$ into consecutive pieces.
- b) If we order the pieces in counterclockwise order on \mathbf{C}_γ starting from ν , then the m -th piece is the $[d]$ -vertex set of the subtree that contains vertex s_{j_m} for any $1 \leq m \leq t$.

We can restate part of Corollary 3.11/(ii) with this definition using the connection between γ_i and Q_i discussed in Lemma 3.9.

Corollary 4.2. *Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$. Then s_{r-1} has CPP on (G, τ) .*

The properties CPP and CICPP are not independent. In fact we have the following lemma.

Lemma 4.3. *Suppose $S' \subseteq S$ and γ is a cycle in S_d . Let G be an S' - $\text{supp}(\gamma)$ bipartite tree. Suppose $s \in S'$. If all the $[d]$ -vertices incident to s have CICPP on (G, γ) , then s has CPP on (G, γ) .*

Proof. Suppose ν_1, \dots, ν_k are the $[d]$ -vertices incident to s . Let Q_1, \dots, Q_k be the subtrees containing ν_1, \dots, ν_k respectively obtained from G by removing s and its incident edges. One sees that it suffices to show that for each $i : 1 \leq i \leq k$, the union of $[d]$ -vertex sets of $Q_{i'}$ with $i' \neq i$ is a consecutive piece on \mathbf{C}_γ . However, this follows from that ν_i has CICPP on (G, γ) since this union is exactly the $[d]$ -vertex set of the tree containing s obtained by deleting the edge $\{s, \nu_i\}$ from G . \square

We now state the main result of this section.

Proposition 4.4. *Suppose $G \in \mathcal{G}_S(d, r; e_1, \dots, e_{r-1})$.*

Then $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^(d, r, \tau; e_1, \dots, e_{r-1})$ if and only if G satisfies the following conditions:*

- (1) G is a tree.
- (2) Any $[d]$ -vertex of G has CICPP on (G, τ) .

Therefore, by Lemma 4.3, if $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^(d, r, \tau; e_1, \dots, e_{r-1})$, we also have the following:*

- (3) Any S -vertex of G has CPP on (G, τ) .

Example 4.5. Let G be the graph shown in Figure 3, which is the bipartite graph associated to the factorization defined in Example 3.2.

From Example 3.10, we see that s_9 has CPP on $(G, (1 \ 2 \ \dots \ 20))$, where the corresponding partition is $\{\{3, 4, 5\}, \{6\}, \{7, 8, \dots, 11\}, \{12\}, \{13, 14, \dots, 20, 1, 2\}\}$.

If we remove the $[d]$ -vertex 19 and all its incident edges, we get three trees T_1 , T_2 and T_3 whose vertex sets are $\{s_2, s_6\} \cup \{14, 15, \dots, 18\}$, $\{s_1, s_3, s_4, s_5, s_7, s_9\} \cup \{1, 2, \dots, 13\}$ and $\{s_8\} \cup \{20\}$, respectively. It is easy to see that the $[d]$ -vertex sets of T_1 , T_2 and T_3 partition the circle $(1 \ 2 \ \dots \ 18 \ 20)$ into consecutive pieces, and these pieces are in counterclockwise order on the circle starting from 19. Moreover,

the S -vertices incident to 19 are s_2, s_6 and s_8 , and satisfy that $s_2 \in T_1$, $s_6 \in T_2$ and $s_8 \in T_3$. Thus 19 has CICPP on $(G, (1\ 2\ \cdots\ 20))$.

The readers can check that all the other S -vertices have CPP on $(G, (1\ 2\ \cdots\ 20))$, and all the other $[d]$ -vertices have CICPP on $(G, (1\ 2\ \cdots\ 20))$.

We will use the following lemma to prove Proposition 4.4.

Lemma 4.6. *Suppose G is an S - $[d]$ bipartite tree. Let ν_0 be a $[d]$ -vertex of G .*

Suppose Q and \bar{Q} are two subtrees of G satisfying: (1) The union of Q and \bar{Q} is G ; (2) ν_0 is the only common vertex of Q and \bar{Q} ; (3) the $[d]$ -vertex set of Q is a consecutive piece on \mathbf{C}_τ and ends with ν_0 when reading clockwise.

Let γ be the cycle obtained by reading the $[d]$ -vertices of Q in clockwise order on \mathbf{C}_τ . Then we have the following:

- (i) *For any $\nu \neq \nu_0$ a $[d]$ -vertex of Q , ν has CICPP on (Q, γ) if and only if ν has CICPP on (G, τ) .*

If we suppose further that $\{s_{j_1} < s_{j_2} < \cdots < s_{j_t}\}$ are the set of S -vertices incident to ν_0 in G , and $s_{j_1}, \dots, s_{j_{t-1}}$ are in Q and s_{j_t} is in \bar{Q} , then

- (ii) *ν_0 has CICPP on (Q, γ) if and only if ν_0 has CICPP on (G, τ) .*

Proof. The labeling of τ does not matter, so we can always relabel so that ν_0 is the size of the $[d]$ -vertex set of Q . Without loss of generality, we assume $\tau = (1\ 2\ \cdots\ d)$ and the $[d]$ -vertex set of Q is $\{1, 2, \dots, \nu_0\}$. So the $[d]$ -vertex set of \bar{Q} is $\{\nu_0, \nu_0 + 1, \dots, d\}$. We also let γ be the cycle $(1\ 2\ \cdots\ \nu_0)$.

- (i) Suppose by removing ν and its incident edges from G , we get trees T_1, \dots, T_t . We can assume T_1 is the tree that contains ν_0 . Let T'_1 be the tree obtained from T_1 by deleting \bar{Q} . One can check that T'_1, T_2, \dots, T_t are the trees we obtain by removing ν and its incident edges from Q .

Suppose ν has CICPP on (Q, γ) . Then the $[d]$ -vertices of T'_1, T_2, \dots, T_t partition $\mathbf{C}_\gamma \setminus \{\nu\}$ into consecutive pieces. Because T'_1 contains ν_0 , the $[d]$ -vertex set of T'_1 is of the form $\{\alpha, \alpha + 1, \dots, \nu_0, 1, 2, \dots, \beta\}$ for some $0 \leq \beta < \alpha \leq \nu_0$, and the other $t - 1$ trees partition $[\beta + 1, \alpha - 1] \setminus \{\nu\}$ into consecutive pieces. However, the $[d]$ -vertices of \bar{Q} are $\{\nu_0, \nu_0 + 1, \dots, d\}$. Hence, the $[d]$ -vertices of T_1 are $\{\alpha, \alpha + 1, \dots, d, 1, 2, \dots, \beta\}$. Therefore, the $[d]$ -vertices of T_1, T_2, \dots, T_t partition $\mathbf{C}_\tau \setminus \{\nu\}$ into consecutive pieces. Moreover, condition b) in the definition of ν having CICPP on (G, τ) can also be verified. Therefore we proved that ν has CICPP on (G, τ) .

By similar arguments we can prove the other direction that if ν has CICPP on (G, τ) , then ν has CICPP on (Q, γ) .

- (ii) Let T_1, \dots, T_t be the subtrees obtained from G by removing ν_0 and its incident edges, where T_m contains s_{j_m} for each $m : 1 \leq m \leq t$. One checks that \bar{Q} is the union of T_t and the edge $\{\nu_0, s_{j_t}\}$ and Q is the union of T_1, \dots, T_{t-1} and edges $\{\{\nu_0, s_{j_m}\}\}_{m=1}^{t-1}$. Hence, T_1, \dots, T_{t-1} are the trees we obtain by removing ν_0 and its incident edges from Q , and the $[d]$ -vertex set of T_t is $\{\nu_0 + 1, \nu_0 + 2, \dots, d\}$. Now it is easy to verify that ν_0 has CICPP on (Q, γ) if and only if ν_0 has CICPP on (G, τ) .

□

Proof of Proposition 4.4. We prove the proposition by induction on r . Suppose $r = 2$. The condition $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ is equivalent to $e_1 = d$. Under this condition,

$\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$ contains only one graph $G_0 = (\{s_1\} \cup [d], \{\{s_1, \nu\}\}_{\nu=1}^d)$, which satisfies (1) and (2). On the other hand, if G satisfies (1) and (2), one sees that $G = G_0$, which is in $\mathcal{G}_S^*(d, r, \tau; e_1, \dots, e_{r-1})$. Furthermore, we have to have $d = e_1$.

Suppose $r_0 \geq 3$ and the proposition holds for any $r < r_0$. We prove the case $r = r_0$. Let $G = (V, E) \in \mathcal{G}_S(d, r; e_1, \dots, e_{r-1})$. For convenience, for each $j : 1 \leq j \leq r-1$, we define the following:

- Let E_j be the set of edges in G that are incident to s_j .
- Let P_j be the “star-shaped” graph whose vertices are s_j and the e_j $[d]$ -vertices incident to s_j , and whose edge set is E_j .

Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and G is the graph associated to a factorization $(\sigma_1, \dots, \sigma_{r-1})$. Then G is a tree by Corollary 3.4. We only need to show (2). Let $k, Q_1, \dots, Q_k, Q_{k+1}, \dots, Q_{e_{r-1}}, \gamma_1, \dots, \gamma_k, \gamma_{k+1}, \dots, \gamma_{e_{r-1}}$ and B_1, \dots, B_k be defined as in Lemma 3.9. By Lemma 3.9/(iii),(iv) and the induction hypothesis, we have Q_i satisfies (1) and (2) for $1 \leq i \leq k$.

For any $i : 1 \leq i \leq k$, we define ν_i to be the $[d]$ -vertex of Q_i that was incident to s_{r-1} and \bar{Q}_i the union of P_{r-1} and $\cup_{i' \neq i} Q_{i'}$. One checks that the union of Q_i and \bar{Q}_i is G and ν_i is the only common vertex of Q_i and \bar{Q}_i . Thus, using these together with Corollary 3.11/(ii), one sees that the hypothesis for (i) of Lemma 4.6 are satisfied by setting $Q = Q_i$, $\bar{Q} = \bar{Q}_i$ and $\gamma = \gamma_i$.

Let ν be an S -vertex of G . Suppose ν is not in $\text{supp}(\sigma_{r-1})$, the set of vertices incident to s_{r-1} . Then ν is in Q_i for some $i : 1 \leq i \leq k$. Since ν has CICPP on (Q_i, γ_i) , by Lemma 4.6/(i), ν has CICPP on (G, τ) . Suppose ν is in $\text{supp}(\sigma_{r-1})$. Then $\nu \in Q_i$ for some $i : 1 \leq i \leq e_{r-1}$. If $i > k$, ν is the only vertex in Q_i and s_{r-1} is the only vertex that is incident to ν . Then ν automatically has CICPP on (G, τ) . Suppose $i \leq k$. Since s_{r-1} is the biggest S -vertex incident to ν , the conclusion follows from Lemma 4.6/(ii) and the fact that ν has CICPP on (Q_i, γ_i) .

Therefore, we proved that if $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$ and $G \in \mathcal{G}_S^*(d, r, \tau, e_1, \dots, e_{r-1})$, then G satisfies (1) and (2).

Suppose G satisfies (1) and (2). Since G is a tree which is connected, by Lemma 3.3, $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Hence, we only need to prove that G is a factorization graph of type $(d, r, \tau; e_1, \dots, e_{r-1})$. For each $j : 1 \leq j \leq r-1$, we define σ_j to be the e_j -cycle obtained by reading $[d]$ -vertices incident to s_j in clockwise order as appeared in \mathbf{C}_τ . It suffices to show that $\sigma_1 \cdots \sigma_{r-1} = \tau$.

We assume the $[d]$ -vertices incident to s_{r-1} are $\nu_1, \dots, \nu_{e_{r-1}}$. Let $Q_1, \dots, Q_{e_{r-1}}$ be the subtrees we obtain by deleting s_{r-1} and its incident edges from G , where Q_i contains ν_i for each i . Since s_{r-1} has CPP on (G, τ) , the $[d]$ -vertex set of Q_i is a consecutive piece on \mathbf{C}_τ containing ν_i . We claim that

- (i) the $[d]$ -vertex set of Q_i is a consecutive piece on \mathbf{C}_τ which ends with ν_i when read in clockwise order, for each $i : 1 \leq i \leq e_{r-1}$.

One sees that it is enough to prove that

- (i') the $[d]$ -vertex set of Q_i does not contain $\tau(\nu_i)$, the number after ν_i on \mathbf{C}_τ in clockwise order, for each i .

We assume to the contrary that for some i , the $[d]$ -vertex set of Q_i contains $\tau(\nu_i)$. Then among the subtrees we obtain by removing ν_i and its incidence edges, the one containing s_{r-1} does not contain the vertex $\tau(\nu_i)$, which contradicts the assumption that ν_i has CICPP on (G, τ) . Therefore, (i') holds and thus (i) holds.

Let m_i be the size of the $[d]$ -vertex set of Q_i for each i . Without loss of generality, we may assume $m_1, \dots, m_k \geq 2$ and $m_{k+1} = \dots = m_{e_{r-1}} = 1$ for some k .

Since all the e_j 's are greater than 1, any Q_i for $k+1 \leq i \leq e_{r-1}$ does not contain any S -vertices. Therefore, each s_j for any $j \in [r-2]$ is in one of Q_1, \dots, Q_k . Let B_i be the set of j 's where s_j is Q_i , for any $i : 1 \leq i \leq k$. We check that Q_i is the union of P_j for all $j \in B_i$ and G is the union of P_{r-1} and $\cup_{i=1}^k Q_i$.

For each $i : 1 \leq i \leq e_{r-1}$, let γ_i be the cycle obtained by reading the $[d]$ -vertex set of Q_i on \mathbf{C}_τ in clockwise order. Because Q_i 's have property (i), by Lemma 3.5 with $\eta = \sigma_{r-1}^{-1}$ and $\mu = \tau$, we have that $\prod_{i=1}^{e_{r-1}} \gamma_i$ is the cycle decomposition of $\tau \sigma_{r-1}^{-1}$.

Moreover, since $\gamma_{k+1}, \dots, \gamma_{e_{r-1}}$ are cycles of length 1, we have $\prod_{i=1}^k \gamma_i = \tau \sigma_{r-1}^{-1}$.

Let $i : 1 \leq i \leq k$. One sees that $Q_i \in \mathcal{G}_S(m_i, \#B_i + 1; (e_j)_{j \in B_i})$. It is clear that Q_i is a tree because G is a tree. We then claim Q_i also satisfies the following:

(ii) Any $[d]$ -vertex of Q_i has CICPP on (Q_i, γ_i) .

We can prove (ii) similarly as we did in the first half of this proof by using Lemma 4.6. We omit the details.

Now by the induction hypothesis, we have that $Q_i \in \mathcal{G}_S^*(m_i, \#B_i + 1, \gamma_i; (e_j)_{j \in B_i})$, which implies that $\prod_{j \in B_i} \sigma_j = \gamma_i$. Since for any $j_1 \in B_{i_1}$ and $j_2 \in B_{i_2}$ with $i_1 \neq i_2$, we have that $\text{supp}(\sigma_{j_1})$ and $\text{supp}(\sigma_{j_2})$ are disjoint, σ_{j_1} and σ_{j_2} commute. Hence,

$$\prod_{j=1}^{r-2} \sigma_j = \prod_{i=1}^k \prod_{j \in B_i} \sigma_j = \prod_{i=1}^k \gamma_i = \tau \sigma_{r-1}^{-1}.$$

Therefore, $\sigma_1 \dots \sigma_{r-1} = \tau$.

Thus, we proved that the proposition holds for $r = r_0$. \square

5. A BIJECTION BETWEEN FACTORIZATION GRAPHS AND MULTI-NODED ROOTED TREES

Let $S = \{s_1 < s_2 < \dots < s_{r-1}\}$ be a set of positive integers disjoint from $\{0, 1, \dots, d\}$. Also, by convention, we set $s_0 = 0$. (So $s_0 < s_1 < \dots < s_{r-1}$.) For convenience, we assume $\tau = (1 \ 2 \ \dots \ d)$. In this section, we will give a bijection between factorization graphs in $\mathcal{G}_S^*(d, r, \tau = (1 \ \dots \ d); e_1, \dots, e_{r-1})$ and multi-noded rooted trees in $\mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$ assuming $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Clearly, such a bijection can be extended to any τ .

We now construct our bijection.

Definition 5.1. Assume $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. For any $G \in \mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$, we have that G is a tree by Corollary 3.4 or Proposition 4.4. We make the $[d]$ -vertex 1 of G a root, and call the resulting rooted tree $G^{\mathcal{R}}$. It is clear that s_i has $e_i - 1$ children in $G^{\mathcal{R}}$, for each $i : 1 \leq i \leq r - 1$.

Recall that labeled multi-noded rooted trees are defined in Definition 2.8. We define $\Phi^{\mathcal{L}}(G) = (M, \mathfrak{l})$ to be the labeled multi-noded rooted tree, where M is in its multi-noded representation obtained from $G^{\mathcal{R}}$ in the following way:

- We make the root 1 of $G^{\mathcal{R}}$ a single-noded vertex, which is the root of $\Phi^{\mathcal{L}}(G)$. We keep the node label 1 and label the single-noded vertex with $s_0 = 0$.
- For each $i : 1 \leq i \leq r - 1$, suppose $\nu_1 < \dots < \nu_{e_i-1}$ are the children of s_i and ν is the parent of s_i in $G^{\mathcal{R}}$. Let s_i be an $(e_i - 1)$ -noded vertex containing

nodes which are labeled by $\nu_1, \dots, \nu_{e_i-1}$ from left to right. Then connect s_i to the node ν .

One sees that $\Phi^{\mathcal{L}}(G) = (M, \mathfrak{l})$ is in $\mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$, where $M \in \mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$. We denote M by $\Phi(G)$.

Example 5.2. Let $d = 20$, $r = 10$, $\tau = (1 \ 2 \ \dots \ 20)$ and G be the graph shown in Figure 3, which is the bipartite graph associated to the factorization defined in Example 3.2. Then $G \in \mathcal{G}_S^*(d, r, \tau; 2, 3, 2, 3, 3, 4, 4, 2, 5)$ and $\Phi^{\mathcal{L}}(G)$ is a labeled multi-noded rooted tree in $\mathcal{LMR}_S(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$. Figure 5 shows the multi-noded representation of $\Phi^{\mathcal{L}}(G)$. After removing labels for the nodes, we get $\Phi(G)$, which is the multi-noded rooted tree shown in Figure 1(b).

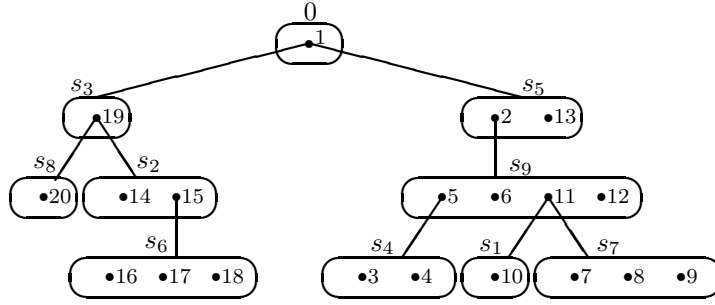


FIGURE 5. A multi-noded rooted tree with labeled nodes.

The main goal of this section is to prove the following proposition.

Proposition 5.3. *Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then Φ gives a bijection from $\mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$ to $\mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$.*

We define $\mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$ to be the set of all labeled multi-noded rooted trees $\Phi^{\mathcal{L}}(G)$ associated to factorization graphs $G \in \mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$. Then

$$\mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1) \subset \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1).$$

The map Φ can be factored into two steps:

$$\begin{array}{ccc} \mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1}) & \xrightarrow{\Phi^{\mathcal{L}}} & \mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1) \\ & \searrow \Phi & \downarrow \text{removing labels of nodes} \\ & & \mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1) \end{array}$$

Hence, Proposition 5.3 follows from the following two lemmas.

Lemma 5.4. *Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then $\Phi^{\mathcal{L}}$ is a bijection from $\mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$ to $\mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$.*

Lemma 5.5. *Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. For any $M \in \mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$, there exists a unique labeling \mathfrak{l} of the nodes of M such that $(M, \mathfrak{l}) \in \mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$.*

Proof of Lemma 5.4. Given any $(M, \mathfrak{l}) \in \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$, we define $\Psi(M, \mathfrak{l})$ to be the S - $[d]$ bipartite graph G whose edge set consists of $\{s, \nu\}$ for which ν is either a node contained in vertex s in M or the parent of s in M . It is clear that G is in $\mathcal{G}_S(d, r; e_1, \dots, e_{r-1})$ and is connected. Then by Lemma 3.3, G is a tree. Hence, $\Psi(M, \mathfrak{l})$ is a map from $\mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$ to the set of S - $[d]$ bipartite trees.

For any $G \in \mathcal{G}_S^*(d, r, (1, \dots, d); e_1, \dots, e_{r-1})$, we have that $\Psi(\Phi^{\mathcal{L}}(G)) = G$. Hence, $\Phi^{\mathcal{L}}$ is injective. Thus, the lemma follows. \square

In order to prove Lemma 5.5, we need to discuss properties of the labeling \mathfrak{l} of any $(M, \mathfrak{l}) \in \mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$. For convenience, we give the following definitions:

Definition 5.6. Given $(M, \mathfrak{l}) \in \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$, and any subgraph M' of M , we denote by $\mathfrak{l}(M')$ the set of labels of the nodes in M' .

For any node ν , we denote by M_ν the subtree of M whose root has the single node ν .

For any vertex s , we denote by M_s the subtree of M rooted at s .

Lemma 5.7. Assume $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Let $(M, \mathfrak{l}) \in \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$. Then $(M, \mathfrak{l}) \in \mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$ if and only if there exist $1 \leq \alpha_\nu \leq \beta_\nu \leq d$ for each node ν of M and $1 \leq \alpha'_j \leq \beta'_j \leq d$ for each vertex s_j of M satisfying the following conditions:

- (i) For any ν a node of M , $\mathfrak{l}(M_\nu) = [\alpha_\nu, \beta_\nu] := \{\alpha_\nu, \alpha_\nu + 1, \dots, \beta_\nu\}$.
- (ii) For any s_j a vertex of M , $\mathfrak{l}(M_{s_j}) = [\alpha'_j, \beta'_j]$.
- (iii) Suppose ν is a node contained in the vertex s_j , and $s_{j_1}, \dots, s_{j_\ell}$ are the vertices connected to ν with $j_1 < \dots < j_k < j < j_{k+1} < \dots < j_\ell$ for some $0 \leq k \leq \ell$. Then $\{\mathfrak{l}(\nu)\}, [\alpha'_{j_1}, \beta'_{j_1}], \dots, [\alpha'_{j_\ell}, \beta'_{j_\ell}]$ partition $[\alpha_\nu, \beta_\nu]$ into consecutive pieces with $\beta'_{j_k} < \dots < \beta'_{j_1} < \mathfrak{l}(\nu) < \beta'_{j_\ell} < \dots < \beta'_{j_{k+1}}$.
- (iv) Suppose s_j is a vertex of M . Let $\nu_1, \nu_2, \dots, \nu_{e_j-1}$ be the nodes in s_j from left to right. Then $[\alpha_{\nu_1}, \beta_{\nu_1}], \dots, [\alpha_{\nu_{e_j-1}}, \beta_{\nu_{e_j-1}}]$ partition $[\alpha'_j, \beta'_j]$ into consecutive pieces with $\beta_{\nu_1} < \dots < \beta_{\nu_{e_j-1}}$.

Proof. Suppose $(M, \mathfrak{l}) \in \mathcal{LMR}_S^*(1, e_1 - 1, \dots, e_{r-1} - 1)$. Then $(M, \mathfrak{l}) = \Phi^{\mathcal{L}}(G)$ for some $G \in \mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$. By Proposition 4.4, G satisfies (2) and (3) of Proposition 4.4. It follows directly that $\mathfrak{l}(M_\nu)$ for any node ν and $\mathfrak{l}(M_{s_j})$ for any vertex s_j are consecutive pieces on the circle \mathbf{C} . Furthermore, since 1 is the label of the node in the root, one sees each consecutive piece is actually a consecutive piece of $[1, d]$. Hence, we can define α_ν, β_ν and α'_j, β'_j such that (i) and (ii) are satisfied.

Let ν be a node of M . If ν is the single node labeled by 1 in the root $s_0 = 0$. Because $s_0 < s_{j_1} < \dots < s_{j_\ell}$, (iii) follows from the fact that the $[d]$ -vertex 1 has CICPP on $(G, (1 \ 2 \ \dots \ d))$. Suppose ν is not in the root. We denote by \bar{M}_ν the tree obtained from M by removing M_ν . Then the fact that ν has CICCP on $(G, (1 \ 2 \ \dots \ d))$ implies that $\mathfrak{l}(M_{s_{j_1}}), \dots, \mathfrak{l}(M_{s_{j_k}}), \mathfrak{l}(\bar{M}_\nu), \mathfrak{l}(M_{s_{j_{k+1}}}), \dots, \mathfrak{l}(M_{s_{j_\ell}})$ are consecutive pieces on \mathbf{C} starting from ν in counterclockwise order. Note that $\mathfrak{l}(\bar{M}_\nu)$ contains node 1, and the union of $\mathfrak{l}(M_{s_{j_1}}), \dots, \mathfrak{l}(M_{s_{j_k}}), \mathfrak{l}(M_{s_{j_{k+1}}}), \dots, \mathfrak{l}(M_{s_{j_\ell}})$ and $\{\mathfrak{l}(\nu)\}$ is $\mathfrak{l}(M_\nu)$. Thus, (iii) follows.

Let $j \in \{0, 1, \dots, r-1\}$. If $j = 0$, (iv) clearly holds. Suppose $j \in [r-1]$. One sees that s_j having CPP on $(G, (1 \ 2 \ \dots \ d))$ implies that $\mathfrak{l}(M_{\nu_1}), \dots, \mathfrak{l}(M_{\nu_{e_j-1}})$ partition

$\mathfrak{l}(M_{s_j})$ into consecutive pieces. Furthermore, when we construct $(M, \mathfrak{l}) = \Phi^{\mathcal{L}}(G)$ from G , we require the labels of the nodes in s_j to be in increasing order from left to right. It follows that $\beta_{\nu_1} < \dots < \beta_{\nu_{e_j-1}}$. Therefore, (iv) holds.

Now we prove the other direction. Suppose $(M, \mathfrak{l}) \in \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$ and there exist $1 \leq \alpha_\nu \leq \beta_\nu \leq d$ for each node ν of M and $1 \leq \alpha'_j \leq \beta'_j \leq d$ for each vertex s_j of M satisfying (i)-(iv). Let Ψ be the map from $\mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$ to the set of $S[d]$ bipartite tress defined in the proof of Lemma 5.4, and define $G := \Psi(M, \mathfrak{l})$. We can reverse the proof in the last two paragraphs to show that (iii) and (iv) imply that G satisfies (2) and (3) of Proposition 4.4. Since G is also a tree, using Proposition 4.4, we conclude that $G \in \mathcal{G}_S^*(d, r, (1 \ 2 \ \dots \ d); e_1, \dots, e_{r-1})$. It is sufficient to show that $\Phi^{\mathcal{L}}(G) = (M, \mathfrak{l})$. However, one checks that for any $(M, \mathfrak{l}) \in \mathcal{LMR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$, $\Phi^{\mathcal{L}}(\Psi(M, \mathfrak{l})) = (M, \mathfrak{l})$ if and only if the following two conditions hold:

- (1) The label of the single node in the root s_0 of M is 1.
- (2) For any $j \in [r - 1]$ the labels of the nodes in s_j are in increasing order from left to right.

However, (1) follows from (iii) by letting ν be the single node in s_0 , and (2) follows from the condition $\beta_{\nu_1} < \dots < \beta_{\nu_{e_j-1}}$ in (iv). \square

Proof of Lemma 5.5. Let $M \in \mathcal{MR}_S(1, e_1 - 1, \dots, e_{r-1} - 1)$. By Lemma 5.7, it is equivalent to prove that there exists a unique choice of a labeling \mathfrak{l} for the nodes of M with set $[d]$, and integers $1 \leq \alpha_\nu \leq \beta_\nu \leq d$ for each node ν of M and integers $1 \leq \alpha'_j \leq \beta'_j \leq d$ for each vertex s_j of M such that (i)-(iv) of Lemma 5.7 are satisfied.

For any vertex s_j , we say it is a *level- m* vertex if it has distance m to the root s_0 . We call a node a *level- m* node if it is inside a level- m vertex. We will describe an algorithm to choose the unique \mathfrak{l} , α_ν, β_ν and α'_j, β'_j . The algorithm will assign values in the order of levels: At step (0), we define α'_0 and β'_0 for the root s_0 ; at step (2m+1) (for $m \geq 0$), we define α_ν and β_ν for all level- m nodes; at step (2m+2) (for $m \geq 0$), we define $\mathfrak{l}(\nu)$ for all level- m nodes, and define α'_j and β'_j for all level- $(m + 1)$ vertices.

- (0) For the root s_0 of M , since $M_{s_0} = M$, the set of labels in M_{s_0} is just $[d]$. Therefore, there is a unique way to choose $\alpha'_0 = 1$ and $\beta'_0 = d$.
- (2m+1) Suppose for any vertex s_j at level- m , α'_j and β'_j are defined.

Let s_j be a vertex at level- m and let $\nu_1, \nu_2, \dots, \nu_{e_j-1}$ be the nodes in s_j from left to right. Let n_i be the number of nodes in M_{ν_i} for each $1 \leq i \leq e_j - 1$. Since α'_j and β'_j are defined already, one sees that there is a unique way to choose $\alpha_{\nu_1}, \beta_{\nu_1}, \dots, \alpha_{\nu_{e_j-1}}, \beta_{\nu_{e_j-1}}$ such that (iv) of Lemma 5.7 is satisfied for s_j :

$$\alpha_{\nu_i} := \alpha'_j + \sum_{t=1}^{i-1} n_t, \quad \beta_{\nu_i} := \alpha'_j - 1 + \sum_{t=1}^i n_t, \quad \forall 1 \leq i \leq e_j - 1.$$

Therefore, in this step, we can define α_ν and β_ν for all the nodes at level- m .

- (2m+2) Suppose for any vertex ν at level- m , α_ν and β_ν are defined.

Let ν be a level- m node contained in vertex s_j , and $s_{j_1}, \dots, s_{j_\ell}$ the vertices connected to ν with $j_1 < \dots < j_k < j < j_{k+1} < \dots < j_\ell$ for some $1 \leq k \leq \ell$. Clearly, $s_{j_1}, \dots, s_{j_\ell}$ are level- $(m + 1)$ vertices. Let

n_1, n_2, \dots, n_ℓ be the number of nodes in $M_{j_1}, M_{j_2}, \dots, M_{j_\ell}$. Since α_ν and β_ν are defined already, one sees that there is a unique way to choose $\mathfrak{l}(\nu), \alpha'_{j_1}, \beta'_{j_1}, \dots, \alpha'_{j_{e_j-1}}, \beta'_{j_{e_j-1}}$ such that (iii) of Lemma 5.7 is satisfied for ν :

$$\begin{aligned} \mathfrak{l}(\nu) &= \alpha_\nu + \sum_{t=1}^k n_t; \\ \alpha'_{j_i} &:= \alpha_\nu + \sum_{t=i+1}^k n_t, \quad \beta_{j_i} := \alpha_\nu - 1 + \sum_{t=i}^k n_t, \quad \forall 1 \leq i \leq k; \\ \alpha'_{j_i} &:= \alpha_\nu + \sum_{t=1}^k n_t + 1 + \sum_{t=i+1}^\ell n_t, \quad \beta_{j_i} := \alpha_\nu + \sum_{t=1}^k n_t + \sum_{t=i}^\ell n_t, \quad \forall k+1 \leq i \leq \ell. \end{aligned}$$

Therefore, in this step, we define labels for all the nodes at level- m and α'_j and β'_j for all the vertices at level- $(m+1)$.

It is easy to see that this algorithm defined the unique solutions to $\mathfrak{l}, \alpha_\nu, \beta_\nu, \alpha'_j, \beta'_j$ that satisfy (i)-(iv) of Lemma 5.5. \square

We proved Lemma 5.4 and 5.5. Hence, Proposition 5.3 follows.

Proof of Theorem 1.4. It is clear that the theorem follows from Corollaries 2.7 and 3.12 and Proposition 5.3. \square

Therefore, as we discussed in the introduction, Theorem 1.2 follows.

REFERENCES

- [1] J. Dénes, *The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs*, Publ. Math. Institute Hung. Acad. Sci. **4** (1959), 63–70.
- [2] I.P. Goulden and D.M. Jackson, *Transitive factorizations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), 51–60.
- [3] I.P. Goulden and S. Pepper, *Labelled trees and factorizations of a cycle into transpositions*, Discrete Math. **113** (1993), 263–268.
- [4] I. Goulden and A. Yong, *Tree-like properties of cycle factorizations*, J. Combin. Theory Ser. A, **98**(1) (2002), 106–117.
- [5] A. Hurwitz, *Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Mathematische Annalen, **39** (1891), 1–60.
- [6] J. Irving, *Minimal transitive factorizations of permutations into cycles*, Canad. J. Math., **61** (2009), 1092–1117.
- [7] F. Liu and B. Osserman, *The irreducibility of certain pure-cycle Hurwitz spaces*, Amer. J. Math., **130** (2008), 1687–1708.
- [8] P. Moszkowski, *A solution to a problem of Dénes: a bijection between trees and factorizations of cyclic permutations*, European J. Combinatorics **10** (1989), 13–16.
- [9] C. M. Springer, *Factorizations, trees, and cacti*, Eighth International Conference on Formal Power Series and Algebraic Combinatorics, University of Minnesota, June 25–29, 1996, 427–438.
- [10] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.
- [11] R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.